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Note: On an unsolved problem about the minimal energies of bicyclic graphs

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Abstract For a simple graph G, the energy $E(G)$ is defined as the sum of the absolute values of all eigenvalues of its adjacency matrix. Let $\mathcal{G}_n$ be the class of connected bipartite graphs of order n with exactly two vertex-disjoint cycles. In the paper [F. Wei, Ordering bipartite graphs by their minimal energies, 2008 International Symposium on Information Science and Engineering (ISISE2008), Dec. 20-22, 2008, 471–474], the author showed an order of minimal energies in $\mathcal{G}_n$. Unfortunately, the energy order of the two graphs in $\mathcal{G}_n$ with the second and third minimal energies is not determined. This short note provides a complete proof for this unsolved problem.

Keywords Minimal energy; bipartite graph; approximate root
AMS subject classifications 05C35; 05C50

1 Introduction

In this paper we are concerned with simple graphs. Let G be such a graph with order n.

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of a graph G. Then the energy of the graph G is

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defined as

$$E(G) = \sum_{i=1}^{n} |\lambda_i|. \tag{1.1}$$

So far, the theory of graph energy is well developed; its details can be found in the book [6] and the latest results in [3, 4, 5, 7]. An important tool of graph energy is the Coulson integral formula

$$E(G) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{1}{x^2} \ln\left[\left(\sum_{i=0}^{\lfloor n/2\rfloor}(-1)^i a_{2i}x^{2i}\right)^2 + \left(\sum_{i=0}^{\lfloor n/2\rfloor}(-1)^i a_{2i+1}x^{2i+1}\right)^2\right]\mathrm{d}x, \tag{1.2}$$

where $a_1, a_2, \ldots, a_n$ are the coefficients of the characteristic polynomial $\phi(G;x)$ of G.

According to (1.2), we can define the *quasi-order* on the energies of graphs as in [6], to be $\prec$ (or $\succ$). That is, $G_1 \prec G_2 \Rightarrow E(G_1) < E(G_2)$.

For convenience, we now introduce some terminology and notations which will be used in the sequel. Let $\mathcal{G}_n$ be the class of connected bipartite graphs of order n with exactly two vertex-disjoint cycles. Let U_n^l be the graph obtained by attaching $n-l$ pendent vertices to a vertex of the cycle C_l. Let $B_{n_1,n_2}^{l_1,l_2}$ be the graph obtained by adding an edge between the vertex of maximal degree in $U_{n_1}^{l_1}$ and the vertex of maximal degree in $U_{n_2}^{l_2}$. Let $G_1, \ldots, G_7$ be the graphs on n vertices as depicted in Figure 1. For terminology and notations not defined here, we refer to the book [1].

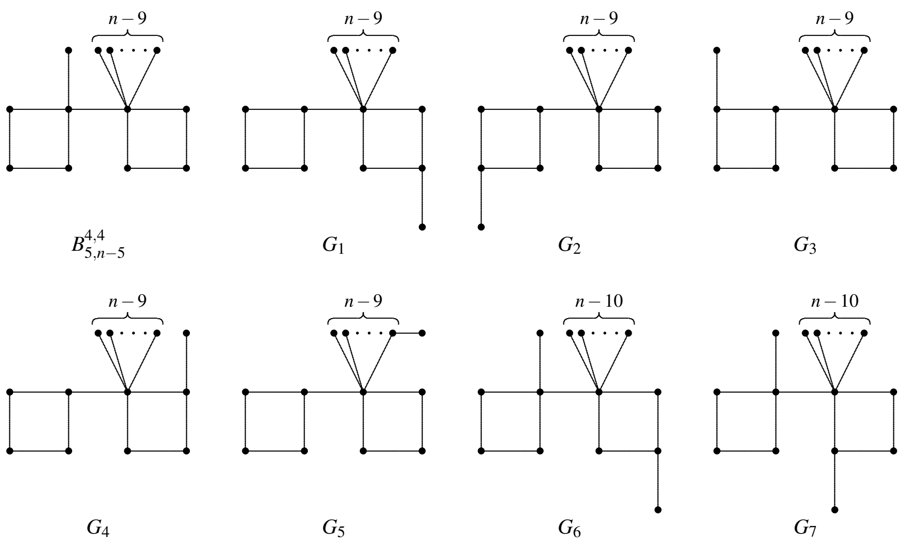

Figure 1. The graphs $B^{4,4}_{5,n-5}$ and G_i, $i = 1, 2, \ldots, 7$.

Recently, Wei [9] obtained the following result.

Theorem 1.1 *If $n \geq 11$, then*

$$E(B^{4,4}_{4,n-4}) < E(B^{4,4}_{5,n-5}) \text{ or } E(G_1) < E(G_2) < E(G_3) < E(G_4) < E(B^{4,4}_{6,n-6}) < E(B^{4,6}_{4,n-4}).$$

Moreover, $E(G_4) < E(G_i)$ ($i = 5,6,7$).

In Theorem 1.1, we know that the energy order of the two graphs $B^{4,4}_{5,n-5}$ and G_1 (see Figure 1) is not determined. In other words, whichever of $B^{4,4}_{5,n-5}$ and G_1 that has the second minimal energy is not determined in $\mathcal{G}_n$.

Furthermore, the author [9] proposed the following remark.

Remark For $n \geq 11$, neither $B^{4,4}_{5,n-5} \prec G_1$ nor $B^{4,4}_{5,n-5} \succ G_1$ and therefore $E(B^{4,4}_{5,n-5})$ and $E(G_1)$ cannot be compared by quasi-order.

In this paper, we settle the above unsolved problem.

2 Second minimal energy

In order to show the main result, we first introduce the following elementary result by direct calculation.

Lemma 2.1

$$\phi(G_1;\lambda) = \lambda^n - (n+1)\lambda^{n-2} + (7n-36)\lambda^{n-4} - (12n-94)\lambda^{n-6},$$
$$\phi(B^{4,4}_{5,n-5};\lambda) = \lambda^n - (n+1)\lambda^{n-2} + (7n-37)\lambda^{n-4} - (12n-96)\lambda^{n-6} + (4n-36)\lambda^{n-8}.$$

We now exhibit our main result as follows.

Theorem 2.2 *G_1 and $B^{4,4}_{5,n-5}$ have the second and the third minimal energies in $\mathcal{G}_n$ for $n \geq 11$, respectively.*

Proof. In order to obtain the assertion, it is sufficient to show $E(B^{4,4}_{5,n-5}) > E(G_1)$. By means of Lemma 2.1, we have

$$\begin{aligned}\phi(G_1;x) &= x^n - (n+1)x^{n-2} + (7n-36)x^{n-4} - (12n-94)x^{n-6} \\ &= x^{n-6}(x^6 - (n+1)x^4 + (7n-36)x^2 - (12n-94)) \\ &\triangleq x^{n-6} f_1(x),\end{aligned}$$

$$\begin{aligned}\phi(B^{4,4}_{5,n-5};x) &= x^n-(n+1)x^{n-2}+(7n-37)x^{n-4}-(12n-96)x^{n-6}\\ &\quad +(4n-36)x^{n-8}\\ &= x^{n-8}(x^8-(n+1)x^6+(7n-37)x^4-(12n-96)x^2\\ &\quad +(4n-36))\\ &\triangleq x^{n-8}f_2(x).\end{aligned}$$

Note that

$$\begin{aligned} f_1(1) &= -6n+58<0 && (n>9),\\ f_1(1.8) &= 0.1824n+0.874624>0 && (n>0),\\ f_1(2) &= -2<0 && (n>0),\\ f_1(\sqrt{n-5}) &= n^2-23n+124>0 && (n>14),\end{aligned}$$

and

$$\begin{aligned} f_2(0.6) &= 0.540544n-6.26505984>0 && (n>11),\\ f_2(1.3) &= -1.114109n+23.8947982<0 && (n>21),\\ f_2(2) &= 4n-52>0 && (n>13),\\ f_2(\sqrt{n-6}) &= -7n^2+112n-432<0 && (n>9),\\ f_2(\sqrt{n}) &= 6n^3-49n^2+100n-36>0 && (n>5).\end{aligned}$$

By the theorem of zero points, we acquire

$$2(1.8+2+\sqrt{n-5})>E(G_1) \qquad \text{for } n>14, \tag{2.1}$$

and

$$2(0.6+1.3+2+\sqrt{n-6})<E(B^{4,4}_{5,n-5}) \qquad \text{for } n>21. \tag{2.2}$$

We have $2(1.8+2+\sqrt{n-5})<2(0.6+1.3+2+\sqrt{n-6})$ for $n\geq 31$. Hence by (2.1) and (2.2), we have $E(G_1)<E(B^{4,4}_{5,n-5})$ for $n\geq 31$. By direct calculation (see Table 1), we have $E(G_1)<E(B^{4,4}_{5,n-5})$ for $11\leq n\leq 30$.

n	$E(B^{4,4}_{5,n-5})-E(G_1)$	n	$E(B^{4,4}_{5,n-5})-E(G_1)$
11	0.71431	12	0.77021
13	0.80429	14	0.82750
15	0.84440	16	0.85732
17	0.86755	18	0.87584
19	0.88277	20	0.88853
21	0.89351	22	0.89780
23	0.90158	24	0.90490
25	0.90785	26	0.91050
27	0.91289	28	0.91505
29	0.91701	30	0.91881

Table 1. The difference between $E(B^{4,4}_{5,n-5})$ and $E(G_1)$.

Therefore we complete the proof. □

References

[1] J. A. Bondy, U. S. R. Murty, Graph Theory, Springer-Verlag, Berlin, 2008.

[2] B. Huo, S. Ji, X. Li, Note on unicyclic graphs with given number of pendent vertices and minimal energy, *Linear Algebra Appl.*, 433 (2010) 1381–1387.

[3] B. Huo, X. Li, Y. Shi, Complete solution to a conjecture on the maximal energy of unicyclic graphs, *European J. Combin.*, 32 (2011) 662–673.

[4] B. Huo, X. Li, Y. Shi, Complete solution to a problem on the maximal energy of unicyclic bipartite graphs, *Linear Algebra Appl.*, 434 (2011) 1370–1377.

[5] J. Li, X. Li, Y. Shi, On the maximal energy tree with two maximum degree vertices, *Linear Algebra Appl.*, 435 (2011) 2272–2284.

[6] X. Li, Y. Shi, I. Gutman, Graph Energy, Springer, New York, 2012.

[7] X. Li, M. Wei, Y. Shi, J. Li, On a conjecture about tricyclic graphs with maximal energy, *MATCH Commun. Math. Comput. Chem.* 72 (2014) 183–214.

[8] X. Li, J. Zhang, On bicyclic graphs with maximal energy, *Linear Algebra Appl.*, 427 (2007) 87–98.

[9] F. Wei, Ordering bipartite graphs by their minimal energies, 2008 International Symposium on Information Science and Engineering (ISISE2008), 20-22, Dec. 2008, 471–474.